Dual geometries and spacetime singularities

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The notion of geometrical duality is discussed in the context of both Brans-Dicke theory and general relativity. It is shown, in some typical cases, that spacetime singularities that arise in usual Riemannian general relativity, may be avoided in its dual representation: Weyl-type general relativity, thus providing a singularity-free picture of the World that is physicaly equivalent to the canonical general relativistic one.

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I. INTRODUCTION

To our knowledge Dicke was first who raised questions about the physical significance of Riemannian geometry in relativity due to the arbitrariness in the metric tensor resulting from the indefiniteness in the choice of units of measure [1,2]. Actually, Brans-Dicke (BD) theory with a changing dimensionless gravitational coupling constant($\hbar = c = 1$): $Gm^2 \sim \phi^{-1}$ (m is the inertial mass of some elementary particle and ϕ is the scalar BD field), can be formulated in two equivalent ways for either m or G could vary with position in spacetime[1]. The choice $G \sim \phi^{-1}$, $m = const.$, leads to the Jordan frame (JF) BD formalism, that is based upon the Lagrangian [1]:

$$L^{BD}[g,\phi] = \frac{\sqrt{-g}}{16\pi}(\phi R - \frac{\omega}{\phi} g^{nm}\nabla_n\phi\nabla_m\phi) + L_{matter}[g] \tag{1.1}$$

where R is the curvature scalar, ω is the BD coupling constant, and $L_{matter}[g]$ is the Lagrangian density for ordinary matter minimally coupled to the scalar field.

For his part, the choice $m \sim \phi^{-\frac{1}{2}}$, $G = const.$, leads to the Einstein frame (EF) BD theory that is founded on the Lagrangian [2]:

$$L^{BD}[\hat{g},\hat{\phi}] = \frac{\sqrt{-\hat{g}}}{16\pi}(\hat{R} - (\omega + \frac{3}{2})\hat{g}^{nm}\hat{\nabla}_n\hat{\phi}\hat{\nabla}_m\hat{\phi}) + \hat{L}_{matter}[\hat{g},\hat{\phi}] \tag{1.2}$$

where now, in the EF metric $\hat{\mathbf{g}}$, the ordinary matter is nonminimally coupled to the scalar field $\hat{\phi} \equiv ln\phi$ through the Lagrangian density $\hat{L}_{matter}[\hat{g},\hat{\phi}]$.

Both JF and EF formulations of BD gravity are equivalent representations of the same physical situation [1] for they belong to the same conformal class, i.e.; EF Lagrangian (1.2) is equivalent to JF Lagrangian (1.1) in respect to the conformal rescaling of the spacetime metric $\mathbf{g} \to \hat{\mathbf{g}} = \phi\mathbf{g}$, which in the coordinate basis can be written as:

$$\hat{g}_{ab} = \phi g_{ab} \tag{1.3}$$

The conformal rescaling (1.3) can be interpreted geometrically as a particular transformation of the units of measure [2]. In this sense, any dimensionless number such as, for example, Gm^2 is invariant under (1.3) so, the observables of the theory, that are always dimensionless numbers[2], are invariant too under this transformation. This means that, concerning experimental observations, both formulations based on varying G (JFBD) and varying m (EFBD) respectively are undistinguishable, i.e., these are physically equivalent.

The same line of reasoning can be applied to the case suggested by Magnano and Sokolowski [4], involving the conformally related Lagrangians:

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[1]Generaly BD theory can be formulated in an infinite number of equivalent ways, for m and G can both vary with position in an infinite number of ways such as to keep $Gm^2 \sim \phi^{-1}$.

[2]A measurement of any dimensional quantity must always represent its ratio to some standard unit, as properly remarked in section II of reference [3])

$$L^{GR}[g,\phi]=\frac{\sqrt{-g}}{16\pi}(\phi R-\frac{\omega}{\phi}g^{nm}\nabla_n\phi\nabla_m\phi)+L_{matter}[g,\phi] \tag{1.4}$$

and

$$L^{GR}[\hat{g},\hat{\phi}]=\frac{\sqrt{-\hat{g}}}{16\pi}(\hat{R}-(\omega+\frac{3}{2})\hat{g}^{nm}\hat{\nabla}_n\hat{\phi}\hat{\nabla}_m\hat{\phi})+\hat{L}_{matter}[\hat{g}] \tag{1.5}$$

where now, unlike the situation we encountered in usual BD gravity, ordinary matter is minimally coupled in the EF (magnitudes with hat), while it is nonminimally coupled in the JF. Both Lagrangians (1.4) and (1.5) represent equivalent pictures of the same theory: general relativity (GR). Actually, it can be seen from (1.5) and, consequently, from the field equations derivable from it:

$$\hat{G}_{ab}=8\pi\hat{T}_{ab}+(\omega+\frac{3}{2})(\hat{\nabla}_a\hat{\phi}\hat{\nabla}_b\hat{\phi}-\frac{1}{2}\hat{g}_{ab}\hat{g}^{nm}\hat{\nabla}_n\hat{\phi}\hat{\nabla}_m\hat{\phi}) \tag{1.6}$$

and

$$\hat{\Box}\hat{\phi}=0 \tag{1.7}$$

with $\hat{G}_{ab}\equiv\hat{R}_{ab}-\frac{1}{2}\hat{g}_{ab}\hat{R}$, $\hat{\Box}\equiv\hat{g}^{nm}\hat{\nabla}_n\hat{\nabla}_m$ and the conservation equation:

$$\hat{\nabla}_n\hat{T}^{na}=0 \tag{1.8}$$

where $\hat{T}_{ab}=\frac{2}{\sqrt{-\hat{g}}}\frac{\partial}{\partial\hat{g}^{ab}}(\sqrt{-\hat{g}}\hat{L}_{matter})$ are the components of the stress-energy tensor for ordinary matter, that the theory given by the Lagrangian (1.5) is just GR with a scalar field as an additional source of gravity. In particular, it can be verified that both weak equivalence principle (WEP) and strong equivalence principle (SEP) hold in this case [5]. We call the theory derivable from (1.5) as EFGR, while its conformally equivalent representation based upon Lagrangian (1.4) we call JFGR.

Now we shall list some features of the JFGR theory. First, we shall present those aspects that constitute its main disadvantages: BD scalar field is nonminimally coupled both to scalar curvature and to ordinary matter so, in particular, the gravitational constant G varies from point to point; $G\sim\phi^{-1}$ and, at the same time, test particles don't follow the geodesics of the geometry. This leads that test particles inertial masses vary from point to point in spacetime in such a way as to preserve the constant character of the dimensionless gravitational coupling constant Gm^2, i.e.; $m\sim\phi^{\frac{1}{2}}$.

In the present paper we shall focus, however, on one aspect of the JFGR that represents an advantage of this formulation in respect to its conformal EF formulation of general relativity; it is respecting spacetime singularities: in this frame (JF) the energy conditions, in general, don't hold so, singularities that can be present in the EFGR may be smoothed out [6] and, in some cases, avoided.

To our knowledge, up to date, only the EF formulation of general relativity (canonical GR) and, consequently, Riemann manifolds with singularities it leads, have been paid attention in the literature. This historical omission is the main motivation for the present work.

The paper has been organized as follows: in Sec. II we discuss on geometrical duality in BD gravity and GR theory. In Sec. III we present the JF formulation of general relativity in detail. Secs. IV and V are aimed at the study of particular solutions to GR theory that serve as illustrations to the conception of geometrical duality previously discussed in Sec. II. For simplicity we shall focus mainly in the value $\omega=-\frac{3}{2}$ for the BD coupling constant. In this case EFGR reduces to canonical Einstein's theory[3]. In particular the Schwarzschild solution is studied in Sec. IV, while flat Friedman-Robertson-Walker (FRW) cosmology for perfect fluid ordinary matter with a barotropic equation of state is studied in Sec. V. Finally, in Sec. VI, conclusions are given.

II. GEOMETRICAL DUALITY

Usual JF formulation of BD gravity is linked with Riemann geometry [1]. It is directly related to the fact that, in JFBD formalism, ordinary matter is minimally coupled to the scalar BD field through $L_{matter}[g]$ in (1.1), leading

[3]For $\omega=-\frac{3}{2}$ in the EF the scalar field is unphysical and doesn't influence the physics in this frame

that point particles follow the geodesics of the Riemann geometry, that is based upon the vector transplantation law: $d\xi^a = -\Gamma^a_{mn}\xi^m dx^n$, and the length preservation requirement: $dg(\xi,\xi) = 0$ where, in the coordinate basis $g(\xi,\xi) = g_{nm}\xi^n\xi^m$, Γ^a_{bc} are the affine connections of the manifold, and ξ^a are the components of an arbitrary vector ξ.

The above postulates of vector transplantation and legth preservation lead that, in Riemann geometry, the affine connections of the manifold coincide with the Christoffel symbols of the metric $\mathbf{g}$:$\Gamma^a_{bc} = \frac{1}{2}g^{an}(g_{nb,c} + g_{nc,b} - g_{bc,n})$. However, under the rescaling (1.3), the above transplantation law is mapped into:

$$d\xi^a = -\hat{\gamma}^a_{mn}\xi^m dx^n \tag{2.1}$$

where $\hat{\gamma}^a_{bc} = \hat{\Gamma}^a_{bc} - \frac{1}{2}(\hat{\nabla}_b\hat{\phi}\delta^a_c + \hat{\nabla}_c\hat{\phi}\delta^a_b - \hat{\nabla}^a\hat{\phi}\hat{g}_{bc})$ are the affine connections of a Weyl-type manifold given by the length transplantation law:

$$d\hat{g}(\xi,\xi) = dx^n\hat{\nabla}_n\hat{\phi}\hat{g}(\xi,\xi) \tag{2.2}$$

In this case the affine connections of the manifold dont coincide with the Christoffel symbols of the metric and one can define metric and affine magnitudes and operators on the Weyl-type manifold.

We get that, under the rescaling (1.3), Riemann geometry with normal behaviour of units of measure changes into a more general Weyl-type geometry with units of measure varying length in spacetime according to (2.2). At the same time, as shown in section I, JF and EF Lagrangians (1.1) and (1.2) for BD theory and (1.4) and (1.5) for GR with an extra scalar field respectively, are connected too by the conformal rescaling of the metric (1.3) (together with the scalar field redefinition $\phi \to \hat{\phi} = ln\phi$). This means that, respecting conformal transformation (1.3), JF and EF formulations of the theory (GR or BD) on the one hand, and Riemann and Weyl-type geometries, on the other, form classes of conformal equivalence. These classes of conformal gravity theories, on the one hand, and conformal geometries, on the other, can be uniquely linked only after coupling of the matter fields to the metric has been specified.

Take, for example, the BD theory. In this case matter minimally couples in the JF so, test particles follow the geodesics of the Riemann geometry in this frame, i.e., JFBD theory is naturally linked with Riemann geometry. This means that EFBD theory, conformally equivalent to JF one, is constrained to be linked with the conformally equivalent to Riemann geometry; Weyl-type geometry. For general relativity with an extra scalar field just the contrary is true. In this case matter minimally couples in the EF, so test particles follow the geodesics of the Riemann geometry precisely in this frame, i.e., EFGR is naturally linked with Riemann geometry and, consequently, JFGR conformally equivalent to EFGR, is linked with Weyl-type geometry.

When the matter part of the Lagrangian is not present, both JFBD theory and GR theory on the one hand, and EFBD and GR theories, on the other, can be interpreted on the grounds of either Riemann or Weyl-type geometry indistinctly, leading to the conclusion that, in this case, both BD theory and general relativity with an extra scalar field coincide. This degeneration of the geometrical interpretation of the theory can be removed only after setting of its matter content.

The following facts:

1. The observables of the theory, being always dimensionless numbers[4], are invariant under the rescaling (1.3), that can be interpreted as a particular units transformation [1,2] (for an exhaustive discussion on the dimensionless nature of measurements see section II of reference [3]).

2. The choice of the unit of length of the geometry is not an experimental issue (for a classical discussion on this subject we refer the reader to [7]).

lead to the conclusion that physical experiment is not sensitive to the rescaling (1.3) and, consequently, the choice of the spacetime geometry is not an experimental issue for, both Riemann geometry naturally linked with JFBD theory (EFGR) and Weyl-type geometry linked with EFBD theory (JFGR), belong to the same equivalence class in respect to the transformation (1.3).

Our line of reasonig leads that a statement such like: 'the JF formulation of BD theory (the EF formulation of GR theory) is the physical one' is devoid of any physical, i.e., experimentally testable meaning, and can be taken only as an independent postulate of the theory. This means that the discussion about which conformal frame is the physical one [4,6,8] is devoid of interest; it is a non-well-posed question.

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[4]Take, for example, the measurement of the energy E of a given physical system. That one really measures in experiments is the number n of times the unit of energy E_0 fits into the quantity being measured: $E = nE_0$, i.e., the dimensionless quantity E/E_0.

A more interesting situation is approached if we take the following postulate: conformal representations of a given classical theory of gravity are physically equivalent. This postulate leads that the geometrical interpretation of a given physical situation through general relativity, BD theory or Scalar-Tensor(ST) theories in general, is not just one unique picture of it, but a whole equivalence class of all conformally related pictures. This fact we call as 'geometrical duality'. In this sense Riemann and Weyl-type geometries, for instance, are dual to each other for they provide different geometrical pictures originating from the same physical situation, that are equally consistent with the observational evidence. The choice of one or the other picture for the interpretation of the given physical effect is a matter of philosophical prejudice or, may be, mathematical convenience.

The word duality is used here in the same context as in [3], i.e., it has only a semantic meaning and has nothing to do with the notion of duality in string theory.

The rest of this paper is based, precisely, upon the validity of the postulate about the physical equivalence of conformal representations of a given classical theory of gravity. Although the notion of geometrical duality can be illustrated in both BD and GR theories and, in general, in ST theories, in what follows we shall illustrate it in GR with an extra scalar field because the issue of conformal equivalence (in respect to transformation (1.3)) has been less intensively studied in general relativity than in ST theories (including BD gravity).

III. JORDAN FRAME GENERAL RELATIVITY

The field equations of the JFGR theory can be derived, either directly from the Lagrangian (1.4) by taking its variational derivatives respect to the dynamical variables or by conformally mapping eqs.(1.6-1.8) back to the JF metric according to (1.3), to obtain:

$$G_{ab} = 8\pi T_{ab} + \frac{\omega}{\phi^2}(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}g^{nm}\nabla_n\phi\nabla_m\phi) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi) \tag{3.1}$$

$$\Box\phi = 0 \tag{3.2}$$

and

$$\nabla_n T^{na} = \frac{1}{2}\phi^{-1}\nabla^a\phi T \tag{3.3}$$

respectively.

The dynamic equation (3.3) leads that, in particular, the free-motion equation of an uncharged, spinless mass point:

$$\frac{d^2x^a}{ds^2} = -\Gamma^a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} - \frac{1}{2}\phi^{-1}\nabla_n\phi(\frac{dx^n}{ds}\frac{dx^a}{ds} - g^{an}) \tag{3.4}$$

does not coincide with the geodesic of the geometry. At the same time the inertial mass of a given elementary particle varies from point to point in spacetime according to: $m = m_0\phi^{\frac{1}{2}}$, where m_0 is some constant. As we pointed out in Sec.II, this is a consequence of the Weyl-type character of the geometry naturally linked with JFGR.

One of the most salient features of the JFGR theory is that it is invariant in form under the conformal rescaling of the metric [5]:

$$g_{ab} \to \tilde{g}_{ab} = \phi^{2\alpha}g_{ab} \tag{3.5}$$

the field redefinition:

$$\tilde{\phi} = \phi^{1-2\alpha} \tag{3.6}$$

and the coupling constant redefinition:

$$\tilde{\omega} = \frac{\omega - 6\alpha(\alpha-1)}{(1-2\alpha)^2} \tag{3.7}$$

with $\alpha \neq \frac{1}{2}$.[5] This can be proved by direct substitution of eqs.(3.5-3.7) in (1.4), (3.1-3.3). In particular Weyl-type geometry linked with JFGR theory is invariant too under these transformations. Actually, both postulates of

[5]The case $\alpha = \frac{1}{2}$ constitute a singularity in the transformations (3.5-3.7)

vector transplantation (2.1) and length transplantation (2.2) are invariant in form under the rescaling (3.5), the field redefinition (3.6) and the coupling constant redefinition (3.7), i.e., respecting symmetry requirements of a field theory, JFGR formalism would be a preferred option compared with EF formulation of general relativity.

It should be stressed that, in this case (JFGR), the full theory is invariant in respect to transformations (3.5-3.7) unlike the situation takes place in JFBD gravity, where the presence of ordinary matter with $T \equiv T^n_n \neq 0$ breaks this symmetry [9].

Another remarkable feature of the JFGR theory is that, in general, the energy conditions don't hold due, on the one hand, to the term with the second covariant derivative of the scalar field in the righthand side of eq.(3.1) and, on the other, to the constant factor (the BD coupling constant) in the second term (the scalar field energy density term) that can take negative values. This way the righthand side of eq.(3.1) may be negative definite leading that some singularity theorems may not hold and, as a consequence, spacetime singularities that can be present in canonical Riemannian GR (given by eqs.(1.5-1.8)), in Weyl-type GR (JFGR) spacetimes may become avoidable[6].

In what follows we shall illustrate this feature of GR theory in some typical situations, for the particular case when the BD coupling constant is taken to be $\omega = -\frac{3}{2}$[7]. In this case, in the EF, the scalar field stress-energy tensor (second term in the righthand side of eq.(1.6)) vanishes, so we recover the canonical Einstein's GR theory with ordinary matter as the only source of gravity. Usual Einstein's GR theory can be approached as well if we set $\phi = const.$.

Although in this frame the scalar field $\hat{\phi}$ which fulfills with the field equation (1.7) acts, in this case ($\omega = -\frac{3}{2}$), as a non interacting (nor with matter nor with curvature), massless, uncharged and spinless, 'ghost' field, i.e.; it is an unphysical field, it influences the physics in the JF, so its functional form in the EF must be taken into account. For $\omega > -\frac{3}{2}$, $\hat{\phi}$ is a physical field in the EF.

In the following section we shall discuss on geometrical duality among singular Schwarzschild (EF) vacuum solution and the corresponding non singular JF solution and, in section V, we shall illustrate this kind of duality for flat, perfect fluid Friedman - Robertson - Walker (FRW) cosmologies. A similar discussion on conformal transformations between singular and non singular spacetimes in the low-energy limit of string theory can be found in [11] for axion - dilaton black hole solutions in $D = 4$ and in [12] for classical FRW axion - dilaton cosmologies[8]. For spurious black hole in the classical approximation see [13].

IV. GEOMETRICAL DUALITY AND SCHWARZSCHILD BLACK HOLE

In this section, for simplicity, we shall interested in the static, spherically symmetric solution to Riemannian general relativity (EFGR) for material vacuum, with $\omega = -\frac{3}{2}$, and its dual; Weyl-type picture (JFGR). In the EF the field equations (1.6-1.8) can be written, in this case, as:

$$\hat{R}_{ab} = 0$$
$$\Box\hat{\phi} = 0 \tag{4.1}$$

The corresponding solution , in Schwarzschild coordinates,looks like($d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$):

$$d\hat{s}^2 = -f dt^2 + f^{-1}dr^2 + r^2 d\Omega^2 \tag{4.2}$$

and

$$\hat{\phi} = q ln f \tag{4.3}$$

where $f = 1 - \frac{2m}{r}$, m is the mass of the point generating the gravitational field, located at the coordinate beginning, and q is an arbitrary real parameter, i.e.; the static, spherically symmetric solution to eq.(4.1) is just the typical

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[6]It should be pointed out that, the fact that, in this frame (JF), the formulation of the theory (BD or GR) does not lead to a well defined energy-momentum tensor for the scalar field (see the r.h.s. of eq.(3.1)), is only apparent [10]. In fact, the terms with the second covariant derivatives of the scalar field contain the connection, and hence a part of the dynamical description of gravity. In ref. [10] a new connection was presented that leads to a canonical form of the scalar field stress-energy tensor in the JF

[7]In BD gravity the case with $\omega = -\frac{3}{2}$ can't be studied because it leads that the field equations of the theory are undefined

[8]References [11,12] were pointed out to us by D. Wands

Schwarzschild black hole solution for vacuum. The corresponding solution for JFGR can be found with the help of the conformal rescaling of the metric (1.3) and the scalar field redefinition $\phi = e^{\hat{\phi}} = f^q$:

$$ds^2 = -f^{1-q}dt^2 + f^{-1-q}dr^2 + \rho^2 d\Omega^2 \tag{4.4}$$

where we have defined the proper radial coordinate $\rho = rf^{-\frac{q}{2}}$. In this case the curvature scalar is given by:

$$R = -\frac{3}{2}\phi^{-2}g^{nm}\nabla_n\phi\nabla_m\phi = -\frac{6m^2q^2}{r^4}(1-\frac{2m}{r})^{q-1} \tag{4.5}$$

The real parameter q labels different spacetimes $(M, g_{ab}^{(q)}, \phi^{(q)})$, so we obtained a class of spacetimes $\{(M, g_{ab}^{(q)}, \phi^{(q)})/q \in \Re\}$ that belong to a bigger class of known solutions [14]. These known solutions are given, however, for an arbitrary value of the coupling constant ω.

We shall outline the more relevant features of the solution given by (4.4). For the range $-\infty < q < 1$ the Ricci curvature scalar (4.5) shows a curvature singularity at $r = 2m$. For $-\infty < q < 0$ this represents a timelike, naked singularity at the origin of the proper radial coordinate $\rho = 0$. We shall drop these spacetimes for they are not compatible with the cosmic censorship conjecture [15]. Situation with $q = 0$ is trivial: in this case conformal transformation (1.1) coincides with the identity transformation that leaves the theory in the same frame. For $q > 0$, the limiting surface $r = 2m$ has the topology of an spatial infinity so, in this case, we obtain a class of spacetimes with two asymptotic spatial infinities[9]: one at $r = \infty$ and the other at $r = 2m$, joined by a wormhole with a throat radius $r = (2+q)m$, or the invariant surface determined by $\rho_{min} = q(1+\frac{2}{q})^{1+\frac{q}{2}}m$. The wormhole is asymmetric under the interchange of the two asymptotic regions ($r = \infty$ and $r = 2m$) [16].

This way, Weyl-type spacetimes dual to the Riemannian Schwarzschild black hole one (line element (4.2)) are given by the class $\{(M, g_{ab}^{(q)}, \phi^{(q)})/q > 0\}$ of wormhole (singularity free)spacetimes. For testing the Weyl-type geometry dual to the Riemann one, however, we need a test particle. Time-like free-motion paths fulfilling eq.(3.4) are complete in the Weyl-type manifold with $2m \leq r \leq \infty$ for $q \geq 2$ so, in this case, time-like test particles don't meet any singularity, i.e., with the help of time-like test particles we can test the absence of singularities (and black holes) in Weyl-type spacetimes of the class $\{M, g_{ab}^{(q)}, \phi^{(q)}/q \geq 2\}$ that are dual to Riemannian (singular) spacetimes $(M, \hat{g}_{ab})$ given by (4.2).

Pictures with and without singularity are different, but equivalent (dual) geometrical representations of the same physical situation. Experimental evidence on the existence of a black hole (enclosing a singularity), obtained when experimental data is interpreted on the grounds of Riemann geometry naturally linked with EFGR theory for $\omega = -\frac{3}{2}$, can serve, at the same time, for evidence on the existence of a wormhole when the same experimental data is interpreted on the grounds of the Weyl-type geometry (linked with JFGR) dual to it.

Although in the present paper we are interested in the particular value $\omega = -\frac{3}{2}$ of the BD coupling constant, it will interesting however to discuss, briefly, what happen for $\omega > -\frac{3}{2}$. In this case there is a physical scalar in the Einstein frame (see eq.(1.6)). The corresponding EF solution to eqs.(1.6) and (1.7) is given by [14]:

$$d\hat{s}^2 = -(1-\frac{2m}{pr})^p dt^2 + (1-\frac{2m}{pr})^{-p}dr^2 + (1-\frac{2m}{pr})^{1-p}r^2 d\Omega^2 \tag{4.6}$$

and

$$\hat{\phi} = q ln(1-\frac{2m}{pr}) \tag{4.7}$$

where $p^2 + (2\omega+3)q^2 = 1$. There is a time-like curvature singularity at $r = \frac{2m}{p}$, so the horizon is shrunk to a point. This means that, in the EF, the validity of the cosmic censorship hypothesis and, correspondingly, the ocurrence of a black hole are uncertain [14].

The JF solution conformally equivalent to (4.6) is given by:

$$ds^2 = -(1-\frac{2m}{pr})^{p-q}dt^2 + (1-\frac{2m}{pr})^{-p-q}dr^2 + R^2 d\Omega^2 \tag{4.8}$$

[9]For $0 < q < 1$ the spatial infinity at $r = \infty$ is Ricci flat, meanwhile, the one at $r = 2m$ is singular. When $q \geq 1$ both spatial infinities are Ricci flat

where we have introduced the proper radial coordinate $R = r(1-\frac{2m}{pr})^{\frac{1-p-q}{2}}$. In this case, when ω is in the range $0 < \omega + 3 < \frac{1}{2(1-p)}$, the Weyl-type JF geometry shows again two asymptotic spatial infinities joined by a wormhole.

Finally we should note that Schwarzschild black hole present in EF (Riemann) geometry and wormhole (singularity free) spacetimes in its dual JF (Weyl-type) geometry are both physical for these geometries can be tested with the help of test particles: when we work in the EF geometry, with the help of test particles that follow the geodesics of the EF metric we can prove the existence of a Schwarzschild black hole while, when working in the JF geometry with the help of test particles that follow free-motion paths given by eq.(3.4) (these don't coincide with the geodesics of the JF geometry) we can prove the absence of singularities (and black holes) in the corresponding Weyl-type spacetimes. We are tempted, however, to give some preference to wormhole spacetimes in the JF geometry for these geometrical objects (JF wormholes) are invariant respecting transformations (3.5-3.7). The EF Schwarzschild black hole, for his part, doesn't possess this symmetry (see section III).

V. GEOMETRICAL DUALITY IN COSMOLOGY

Other illustrations to the notion of geometrical duality come from cosmology. In the EF, the FRW line element for flat space can be written as:

$$d\hat{s}^2 = -dt^2 + \hat{a}(t)^2(dr^2 + r^2 d\Omega^2) \tag{5.1}$$

where $\hat{a}(t)$ is the EF scale factor. Suppose the universe is filled with a perfect-fluid-type matter with the barotropic equation of state (in the EF): $\hat{p} = (\gamma - 1)\hat{\rho}$, $0 < \gamma < 2$. Taking into account the line element (5.1) and the barotropic equation of state, the field equation (1.6) can be simplified to the following equation for determining the EF scale factor:

$$(\frac{\dot{\hat{a}}}{\hat{a}})^2 = \frac{8\pi}{3}\frac{(C_2)^2}{\hat{a}^{3\gamma}} \tag{5.2}$$

while, after integrating eq.(1.7) once, we obtain for the EF scalar:

$$\dot{\hat{\phi}} = \frac{C_1}{\hat{a}^3} \tag{5.3}$$

where C_1 and C_2 are arbitrary integration constants. Solution to eq.(5.2) is found to be:

$$\hat{a}(t) = (\sqrt{6\pi}\gamma C_2)^{\frac{2}{3\gamma}} t^{\frac{2}{3\gamma}} \tag{5.4}$$

while, integrating eq.(5.3) gives:

$$\hat{\phi}^{\pm}(t) = \hat{\phi}_0 \pm \frac{C_1}{(\sqrt{6\pi}\gamma C_2)^{\frac{2}{\gamma}}} \frac{t^{1-\frac{2}{\gamma}}}{1-\frac{2}{\gamma}} \tag{5.5}$$

The JF scale factor $a^{\pm}(t) = \hat{a}(t) exp[-\frac{1}{2}\hat{\phi}^{\pm}(t)]$ is given by the following expression:

$$a^{\pm}(t) = \frac{(\sqrt{6\pi}\gamma C_2)^{\frac{2}{3\gamma}}}{\sqrt{\phi_0}} t^{\frac{2}{3\gamma}} exp[\mp \frac{C_1}{2(\sqrt{6\pi}\gamma C_2)^{\frac{2}{\gamma}}} \frac{t^{1-\frac{2}{\gamma}}}{1-\frac{2}{\gamma}}] \tag{5.6}$$

The proper time t in the EF and τ in the JF are related through:

$$(\tau - \tau_0)^{\pm} = \frac{1}{\sqrt{\phi_0}} \int exp[\pm \frac{\gamma C_1 t^{1-\frac{2}{\gamma}}}{2(\sqrt{6\pi}\gamma C_2)^{\frac{2}{\gamma}}(2-\gamma)}] dt \tag{5.7}$$

For big t ($t \to +\infty$) this gives:

$$(\tau - \tau_0)^{\pm} \approx \frac{C_1 t}{2\sqrt{\phi_0}(\sqrt{6\pi}\gamma C_2)^{\frac{2}{\gamma}}} \tag{5.8}$$

so $t \to +\infty$ implies $\tau \to +\infty$ for both '+' and '-' branches of our solution, given by the choice of the '+' and '-' signs in eq.(5.5).

For $t \to 0$, the r.h.s. of eq.(5.7) can be transformed into:

$$-\frac{\gamma}{\sqrt{\phi_0}}(2-\gamma)\int \frac{exp[\pm Ax]}{x^{\frac{2}{2-\gamma}}}dx \tag{5.9}$$

where we have defined $x \equiv t^{1-\frac{2}{\gamma}}$ and $A \equiv \frac{\gamma C_1}{2(\sqrt{6\pi}\gamma C_2)^{\frac{\gamma}{2}}(2-\gamma)}$. If we take the '-' sign in the exponent under integral (5.9) then, for $t \to 0$ ($x \to \infty$), $\tau \to \tau_0$. If we take the '+' sign, for his part, integral (5.9) diverges for $t \to 0$ so $\tau \to -\infty$ in this last case.

This leads that, in the '-' branch of our solution, the evolution of the universe in the JF is basically the same as in the EF: the flat FRW perfect-fluid-filled universe evolves from a cosmological singularity at the beginning of time $t = 0$ ($\tau = \tau_0$ in the JF), into an infinite size universe at the infinite future $t = +\infty$ ($\tau = +\infty$ in the JF). It is the usual picture in canonical general relativity where the cosmological singularity is unavoidable.

In the '+' branch of the solution, however, the JF flat FRW perfect-fluid-filled universe evolves from an infinite size at the infinite past ($\tau = -\infty$) into an infinite size at the infinite future ($\tau = +\infty$) through a bounce at $t^* = [\frac{3}{4}\frac{\gamma C_1}{(\sqrt{6\pi}\gamma C_2)^{2\gamma}}]^{\frac{\gamma}{2-\gamma}}$ where it reaches its minimum size: $a^* = \frac{1}{\sqrt{\phi_0}}[\sqrt{\frac{3}{32\pi}}\frac{C_1}{C_2}e]^{\frac{2}{3(2-\gamma)}}$, i.e., the JF universe is free of the cosmological singularity, unlike the EF picture where the cosmological singularity is unavoidable. The more general case of arbitrary $\omega > -\frac{3}{2}$ is studied in [17].

VI. CONCLUSION

The main conclusion to be drawn in the light of the viewpoint developed here is that JF and EF representations of a given gravity theory (GR, BD or ST in general), are equivalent rather than alternative to each other. It is not a merely formal, mathematical equivalence: concerning experimental evidence one can not distinguish between both formalisms, for the observables of the theory, being dimensionless numbers, are invariant in respect to the conformal transformation of units (1.3), generating the equivalence between the different Lagrangians and, at the same time, between the Riemann and the Weyl-type geometries linked with them. Respecting symmetry requirements, however, JFGR may be preferred, for this formulation of general relativity possesses a higher degree of symmetry given by its invariance in respect to transformations (3.5-3.7), that are not merely spacetime diffeomorphisms.

It has been shown in some typical situations, that spacetime singularities that are almost unavoidable in usual Riemannian general relativity (EFGR), in Weyl-type GR may become avoidable. It is, however, at the cost of abandoning the usual (and intuitively more simple) Riemannian geometrical setting for the interpretation of the experimental evidence. Take a flat perfect-fluid-filled universe (Sec.V) as an illustration to this: in the Riemannian geometrical setting (EFGR) there is a cosmological (global) singularity in the past ($t = 0$). In this interpretation, units of measure preserve length when transported from one point to another. In particular, the inertial masses of elementary particles are constant over the manifold, leading to a more transparent (though phylosophically in contradiction with the Mach principle) representation of our world.

In its dual, Weyl-type geometrical setting given by the JFGR, for his part, this universe evolves from an infinite size at $t = 0$ (infinite past in terms of the proper time τ), to an infinite size at the infinite future, through a minimum size at some intermediate time. In this case the universe is free of the cosmological singularity (that is very desirable), however, units of measure change length from point to point. In particular, the inertial mass of an elementary particle varies according to: $m \sim e^{-\frac{B}{2\sqrt{t}}}$, so it evolves from $m = 0$ at the infinite past, to a constant value $m = m_0\sqrt{\phi_0}$ at the infinite future. This provides a less transparent geometrical picture of the universe. The loss of transparence in the JF formulation of general relativity, should be weighed, however, against the inevitability of spacetime singularities in canonical EFGR.

The advantages of the viewpoint presented in the present paper are clear when, in at least one of the conformally equivalent representations, the spacetime singularities vanish. Although one can work in anyone of the conformally equivalent frames, nevertheless, when one approaches the singularity one is constrained to work, precisely, in that conformal frame where the singularity vanishes, in order to get a physically meaningfull description of the world.

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